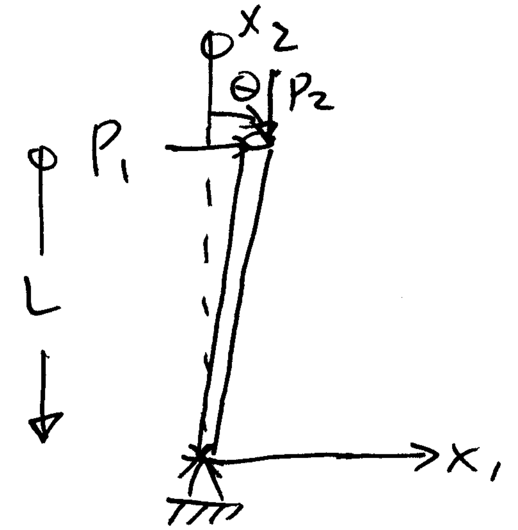
PAL
4/4/08

Unified Engineering Problem Set
Week #10 Spring, 2008

SOLUTIONS

M10.1 Begin with general configuration:



(a) The rod can deform along its length proportional to its stiffness

$$\Delta L = P_2 / k_L$$

with $k_L = EA/L$ as defined

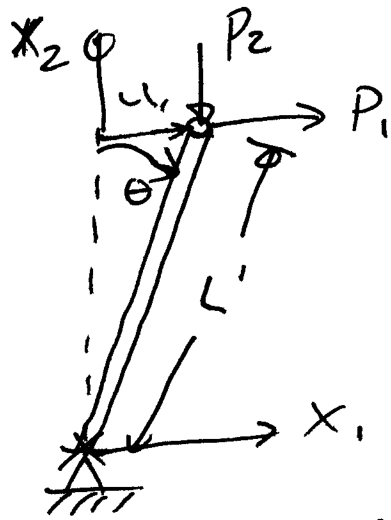
Thus the length, L', at any time is:

$$L' = L - \Delta L \Rightarrow$$

$$L' = L - \frac{P_2}{EA/L} = L\left(1 - \frac{P_2}{EA}\right)$$

This comes with positive ΔL and P_2 defined as being in negative x_2 direction

So this "redraws" the configuration as:



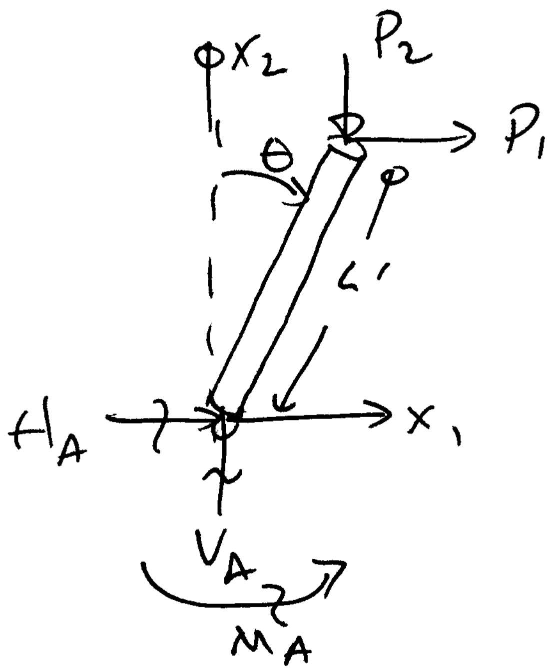
The distance u_1 is $L' \sin\theta$
and again assuming small deformations

$$\Rightarrow u_1 = L'\theta$$

The distance in the x_2-direction is $L' \cos\theta$

$$\Rightarrow \ldots = L'$$

Now draw the Free Body Diagram and take equilibrium:



$\sum F_{x_1} = 0 \ \overset{+}{\rightarrow} \ \Rightarrow P_1 + H_A = 0 \ \Rightarrow H_A = -P_1$

$\sum F_{x_2} = 0 \ \uparrow + \ \Rightarrow V_A - P_2 = 0 \ \Rightarrow V_A = +P_2$

$\sum M_A = 0 \ \circlearrowleft + \Rightarrow -P_1 L' - P_2 L' \theta + \underbrace{k_T \theta}_{} = 0$
(origin)

acts in opposite direction from $+\theta$
$+\theta$ is CW $\Rightarrow$ moment is CCW $\Rightarrow +$

using the latter equation:

$$\theta \, (k_T - P_2 L') = P_1 L'$$

and thus:

$$\theta \left(\frac{k_T - P_2 L'}{L'} \right) = P_1$$

giving

$$\theta \left(\frac{k_T}{L'} - P_2 \right) = P_1$$

use the expression for L' to get:

$$\boxed{\theta\left[\frac{k_T}{L\left(1-\frac{P_2}{EA}\right)}-P_2\right]=P_1}$$

Basic equation governing rotational angle, θ

(b) This expression is basically the form for the effective stiffness K_{eff}:

$$\theta\, K_{eff} = P_1$$

$$\Rightarrow \quad K_{eff} = \frac{k_T}{L\left(1-\frac{P_2}{EA}\right)} - P_2$$

and when $K_{eff} = 0$, instability occurs so the basic equation is:

$$\frac{k_T}{L\left(1-\frac{P_2}{EA}\right)} - P_2 = 0 \qquad (*)$$

working this gives

$$\frac{1}{L\left(1-\frac{P_2}{EA}\right)}\left[k_T - P_2 L + P_2^2\frac{L}{EA}\right] = 0$$

Within the brackets is a quadratic in the applied axial load, P_2. Solve this to find the critical value of P_2 for the instability: $P_{2_{cr}}$

first multiply the expression by $\frac{EA/L}{EA/L}$:

$$\Rightarrow \frac{1}{EA - P_2}\left[P_2^2 - P_2 EA + k_T \frac{EA}{L}\right] = 0$$

solve the quadratic to be zero via:

$$P_{2_{cr}} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

with: $a = 1$
$b = -EA$
$c = k_T \frac{EA}{L}$

$$\Rightarrow \boxed{P_{2_{cr}} = \frac{EA \pm \sqrt{(EA)^2 - 4k_T \frac{EA}{L}}}{2}}$$

expression for critical value of P_2

(Note: can not have $P_2 = EA$ since the term $\frac{1}{EA - P_2}$ would approach ∞)

$\Rightarrow$ That would need to be considered

(c) Now use the ratio:

$$n = EA/(k_T/L)$$

$$\Rightarrow EA = n\frac{k_T}{L}$$

use this in the expression for P_{2cr}:

$$P_{2cr} = \frac{n\frac{k_T}{L} \pm \sqrt{n^2\left(\frac{k_T}{L}\right)^2 - 4n\left(\frac{k_T}{L}\right)^2}}{2}$$

$$\Rightarrow P_{2cr} = \frac{k_T}{2L}\left(n \pm \sqrt{n^2 - 4n}\right)$$

Now it can be seen that to get a real value, must have:

$$n^2 - 4n \geq 0$$

$$\Rightarrow n(n-4) \geq 0$$

$$\Rightarrow n \geq 4$$

→ Behavior: below this value, one gets collapse as the rod is not stiff enough

→ Now consider limits on the values...

for $n = 4$:

$$P_{2cr} = \frac{k_T}{2L}(4) = 2\frac{k_T}{L}$$

→ Behavior: twice the value for the case treated in lecture of infinite stiffness for $n = \infty$

→ There are two behaviors depending upon how ∞ is approached

Case ① For the case of $-\sqrt{\ }$, one gets a look at $EA \to \infty$:

$$P_{2cr} = \frac{k_T}{2L}\left(n - \sqrt{n^2 - 4n}\right)$$

for this case one must look at the residual from $\sqrt{n^2-4n}$ and this is $(n-2)$

So this becomes:

$$P_{2cr} = \frac{k_T}{2L}(2) = \frac{k_T}{L}$$

→ Behavior: Same as value derived in lecture. One can also look at equation (*) and see that if $EA \to \infty$, then $\frac{P_2}{EA} \to 0$ so

$$\left.\begin{aligned} &\frac{k_T}{L} - P_2 = 0 \\ \Rightarrow\ &P_2 = \frac{k_T}{L} \end{aligned}\right\} \text{ as in lecture}$$

Case ② For the case of $+\sqrt{\ }$, look at

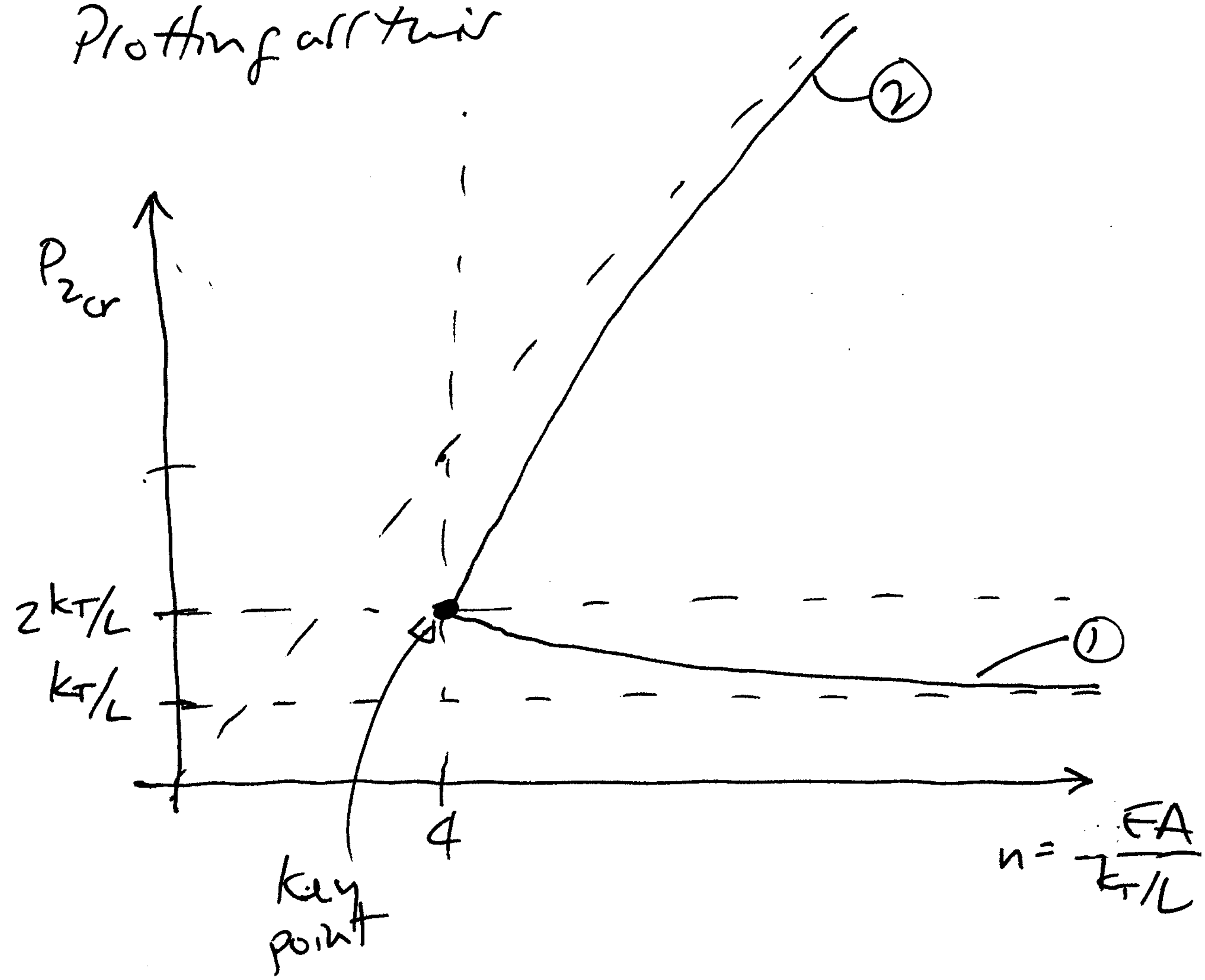
$$\frac{AE}{k_T/L} \to \infty$$

Now: $P_{2cr} = \frac{k_T}{2L}\left(n + \sqrt{n^2}\right)$

$\Rightarrow P_{2cr} = \frac{k_T}{L} n$

$\rightarrow$ <u>Behavior</u>: An asymptotic line proportional to k_T/L

Plotting all this



And always

$P_{cr} > \frac{k_T}{L}$

$P_{cr} < EA$

M10.2

General configuration:

Use a general structural property characterization:

Cross-Section Area = A
Moment of Inertia = I
Modulus = E
End Load = P (related to pressure and Area)

This allows one to characterize the behavior independent of the specifics of the cross-section. This can be grabbed after determining the expression for P_{cr} etc. in terms of A, I, and P by expressing A, I, and P in terms of the details.

Start with the basic governing equation:

$$\frac{d^2 u_3}{dx_1^2} + \frac{P}{EI} u_3 = 0 \qquad (1)$$

with the general homogeneous solution:

$$u_3 = A \sin\left(\sqrt{\frac{P}{EI}}\, x_1\right) + B\left(\cos\sqrt{\frac{P}{EI}}\, x_1\right) + C + Dx_1 \quad (2)$$

Now look at the boundary conditions.

- At the clamped end $(x_1 = 0)$: $u_3 = 0$

$$\frac{du_3}{dx_1} = 0$$

- At the free end with applied load $(x_1 = L)$:

[find from the notes]

$$M = 0 \Rightarrow \frac{d^2u_3}{dx_1^2} = 0$$

$$S = -P\frac{du_3}{dx_1} \text{ and } S = EI\frac{d^3u_3}{dx_1^3}$$

So the condition is:

$$\frac{d^3u_3}{dx_1^3} = -\frac{P}{EI}\frac{du_3}{dx_1}$$

→ To facilitate writing the solution, let us represent: $\lambda = \sqrt{\frac{P}{EI}}$

(Note: A bit different than original $\lambda^2 = -\frac{P}{EI}$)

So (1) becomes:

$$\frac{d^2 u_3}{dx_1^2} + \lambda^2 u_3 = 0 \qquad (1)$$

and (2) becomes:

$$u_3 = A \sin \lambda x_1 + B \cos \lambda x_1 + C + D x_1 \qquad (2)$$

To use the boundary conditions in the basic solution, need derivatives of (2). So:

$$\frac{du_3}{dx_1} = \lambda A \cos \lambda x_1 - \lambda B \sin \lambda x_1 + D$$

$$\frac{d^2 u_3}{dx_1^2} = -\lambda^2 A \sin \lambda x_1 - \lambda^2 \cos \lambda x_1.$$

$$\frac{d^3 u_3}{dx_1^3} = -\lambda^3 A \cos \lambda x_1 + \lambda^3 B \sin \lambda x_1$$

Now apply each of the 4 Boundary Conditions to get 4 equations:

@ $x_1 = 0$, $u_3 = 0 \Rightarrow B + C = 0 \qquad (3)$

@ $x_1 = 0$, $\frac{du_3}{dx_1} = 0 \Rightarrow \lambda A + D = 0 \qquad (4)$

@ $x_1 = L$, $\frac{d^3 u_3}{dx_1^3} = -\underset{\text{as expressed}}{\lambda^2} \frac{du_3}{dx_1}$

$$\Rightarrow -\lambda^3 A\cos\lambda L + \lambda^3 B\sin\lambda L$$
$$= -\lambda^3 A\cos\lambda L + \lambda^3 B\sin\lambda L + \lambda^2 D \quad (5)$$

$$\Rightarrow \boxed{D = 0} \quad (5)$$

and @ $x_1 = L$, $\dfrac{d^2 u_3}{dx_1^2} = 0$

$$\Rightarrow -\lambda^2 A\sin\lambda L - \lambda^2 B\cos\lambda L = 0$$

giving: $A\sin\lambda L + B\cos\lambda L = 0 \quad (6)$

→ Now manipulate these equations:

from (4): $\lambda A = -D$

$$\Rightarrow \lambda A = 0$$
$$\Rightarrow \boxed{A = 0}$$

from (6):

$$B\cos\lambda L = 0$$

This gives 2 possible solutions:

$B = 0$ (trivial .... $u_3 = 0$)

or

$$\cos\left(\sqrt{\frac{P}{EI}}\, L\right) = 0$$

$$\Rightarrow \sqrt{\frac{P}{EI}}\, L = \frac{n\pi}{2} \quad \text{(for n odd)}$$

So:

$$\boxed{P = \frac{n^2\pi^2}{4L^2} EI} \quad \text{(for n odd)}$$

The lowest load occurs for $n = 1$

$$\boxed{P_{cr} = \frac{\pi^2 EI}{4L^2}}$$

buckling load

In general, the form for P_{cr} is:

$$P_{cr} = c\,\frac{\pi^2 EI}{L^2}$$

where c = coefficient of (edge) fixity

so, in this case: $c = 1/4$

To get the mode, go to equation (3) giving:

$$B + C = 0$$

$$\Rightarrow B = -C$$

Use this in the expression for u_3 of equation (2), along with the other results gives:

$$u_3 = B\left[\cos\left(\sqrt{\frac{n^2\pi^2 EI/4L^2}{EI}}\, x_1\right) - 1\right]$$

$$\Rightarrow \quad u_3 = B\left[\cos\left(\sqrt{\frac{n^2\pi^2}{4L^2}}\, x_1\right) - 1\right]$$

forward this gives:

$$u_3 = B\left[\cos\left(\frac{n\pi}{2L} x_1\right) - 1\right]$$

for the case of $n=1$ and P_{cr}:

$$\boxed{u_3 = B\left[\cos\left(\frac{\pi}{2L} x_1\right) - 1\right]}$$

<u>buckling mode</u>

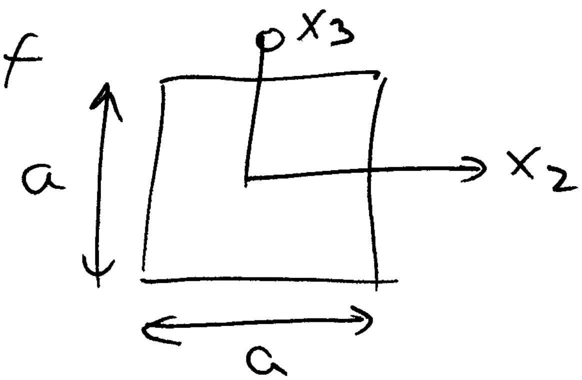
with

$$\boxed{P_{cr} = \frac{\pi^2 EI}{4L^2}}$$

<u>buckling load</u>

Now look at each of the two cases:

(a) Cross-Section of a square



So: $A = a^2 \Rightarrow P = p a^2$

$$I = \frac{bh^3}{12} = \frac{a^4}{12}$$

The buckling mode stays the same

$$u_3 = B\left[\cos\left(\frac{\pi}{2L}x_1\right) - 1\right]$$

but it occurs at a load specific to the configuration:

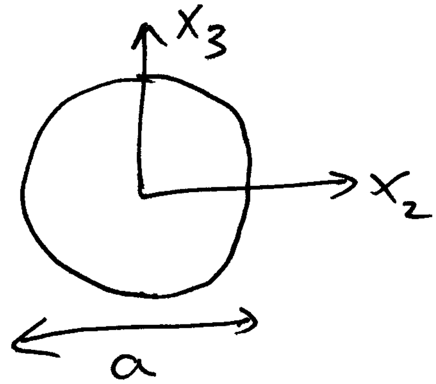
$$P_{cr} = p_{cr}\, a^2 = \frac{\pi^2 E \frac{a^4}{12}}{4L^2}$$

critical pressure

$$\Rightarrow \boxed{p_{cr} = \frac{\pi^2 E a^2}{48 L^2}}$$

critical pressure for square cross-section

(b) Cross-section of a circle



radius $= \frac{a}{2}$

$$\Rightarrow A = \pi\left(\frac{a}{2}\right)^2 = \frac{\pi a^2}{4}$$

$$\Rightarrow P = p\,\frac{\pi a^2}{4}$$

$$I = \frac{\pi r^4}{4} \Rightarrow I = \frac{\pi}{4}\left(\frac{a}{2}\right)^4$$

giving: $I = \frac{\pi a^4}{64}$

again, the buckling mode stays the same

$$u_3 = B\left[\cos\left(\frac{\pi}{2L} x_1\right) - 1\right]$$

but it occurs at a load specific to this cross-section:

$$P_{cr} = p_{cr} \frac{\pi a^2}{4} = \frac{\pi^2 E \frac{\pi a^4}{64}}{4L^2}$$

$$\Rightarrow \boxed{p_{cr} = \frac{\pi^2 E a^2}{64 L^2}}$$

critical pressure for circular cross-section